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VALUATION OF CONVEXITY RELATED INTEREST RATE DERIVATIVES

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Abstract:
We investigate valuation of derivatives with payoff defined as a nonlinear though close to linear function of tradable underlying assets. Interest rate derivatives involving Libor or swap rates in arrears, *i.e*. rates paid at wrong time, are a typical example. It is generally tempting to replace the future unknown interest rates with the forward rates. We show rigorously that indeed this is not possible in the case of Libor or swap rates in arrears. We introduce formally the notion of linear plain vanilla derivatives as those that can be replicated by a finite set of elementary operations and show that derivatives involving the rates in arrears are not (linear) plain vanilla. We also study the issue of valuation of such derivatives. Beside the popular convexity adjustment formula, we develop an improved two or more variable adjustment formula applicable in particular on swap rates in arrears. Finally, we get a precise fully analytical formula based on the usual assumption of log-normality of the relevant tradable underlying assets applicable to a wide class of convexity related derivatives. We illustrate the techniques and different results on a case study of a real life controversial exotic swap.

Keywords: interest rate derivatives, Libor in arrears, constant maturity swap, valuation models, convexity adjustment.

JEL Classification: C13, E43, E47, G13

1. Introduction

We consider European type financial derivatives that are defined as a one or a finite set of payments in specified currencies at specified times, where each payment is uniquely determined at the time it is to be paid as a function of a finite set of already known prices of the underlying assets. Forward transactions, forward rate agreements, swaps, and European options belong to this category. Many forward or swap like instruments can be simply valued using the principle replacing future unknown prices and rates by the forward prices and rates implied by the current market quotes and discounting the resulting fixed cash flow with the risk-free interest rates. This works well for many derivative contracts including Forward Rate Agreements or Interest Rates Swaps. The future interest rates (Libor) can be replaced by the forward rates for the valuation purposes. However, it turns out that this principle is not exactly valid in the case the rates are paid at a "*wrong*" time or in a "*wrong*" currency like in the case of Libor in arrears (*i.e.,* Libor paid at the beginning and not at the end of the interest rate period for

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which it is quoted) or Quanto swaps (where the Libor quotes are taken in one currency but paid in a different currency). Many practitioners still use the forward rate principle as a good approximation for valuation of such products, while others use some kind of a convexity adjustment formula. However, one may still ask the question why the rates paid at a wrong time could not be somehow transferred, *e.g.* using forward discount factors, to the right payment time? Another question is whether and why the popular convexity adjustment formula is correct and how far it is from the best valuation (if there is any)? We will show elementarily that the answer to the first question is no. Then we will give an overview of adjustment formulas used in practice and propose a new one arguing that it is more precise. The impact of different adjustment formulas will be illustrated on a real life example.

2. An Exotic Convexity Related Cross Currency Swap – A Case Study

In March 2003 a large Czech city[1] entered a cross currency swap with a bank intended to hedge the currency and interest rate risk of fix coupon bonds issued in EUR. Details of the transaction are given in Table 1.

When the City Assembly and its Finance Committee have been informed about details of the transaction some of the members questioned the complex and for the needs of the City inappropriate structure of the swap as well as its market parameters. Indeed the first estimates have shown that the market value of the transaction could be quite negative from its very inception. This led to a controversy between the proponents and critics of the transaction.

One of the arguments of the swap proponents was the statement that the only way how to really determine whether the swap was profitable or loss-making would be to wait until its very maturity (*i.e.* 10 years) and then to add up all the cash flows. A resolution in this sense has been even approved by the Controlling Committee, which has investigated various aspects of the transaction and of the bond issue. Even though such a conclusion is fundamentally wrong there is some wisdom in it in the sense that determination of a precise market value at the start and during the life of the swap is indeed a difficult task obscured by a multitude of possible valuation methods and insufficient market data.

Another line of argumentation of the swap supporters has been the statement that the unknown float component of the swap payments, the spread = $IRS_{10} - IRS_2$ defined as the difference between the 10-year and 2-year swap rates quoted at the time of the annual payments in the years 4-10, could be estimated as the average from the past which happened to be around 1.5 %. Hence if the future unknown spreads are replaced by 1.5 % the interest rate paid by the city is estimated at 4.05 %, which is less than the rate 4.25 % paid by the bank. Even though such a valuation method is again fundamentally wrong (recalling the notorious statement saying that past performance is not a guarantee of future profits) it is quite appealing to the laic public.

1 The counterparties of the swap were the City of Prague and Deutsche Bank A.G., Prague Branch. The information has been made public through an information paper provided to the Prague City Assembly.

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Table 1
Details of the Cross Currency Swap, March 2003

	Date/Period	Counterparty A (The City) pays:	Counterparty B (The Bank) pays:
Initial Exchange	19/3/2003	EUR 168 084 100	CZK 5 375 527 500
Fixed Amounts	Annually	4.25 % from the amount of EUR 170 000 000 in the Act/Act Day Count Convention	
Float Amounts	Annually, years 1-3		3.95 % from the amount of CZK 5 389 000 000 in the Actual/360 Day Count Convention
	Annually, years 4-10		(5.55 % - spread) from the amount of CZK 5 389 000 000 in the Actual/360 Day Count Convention, where the Spread is calculated as the difference between the 10-year swap rate minus 2-year swap rate quoted by reference banks 2 business days before the payment
Final Exchange	19/3/2013	CZK 5 389 000 000	EUR 170 000 000

The critics of the swap have on the other hand obtained a specialized consulting firm valuation according to which the market value of the swap using the trade date rates has been CZK -262 million, *i.e.* quite distant from a normal level corresponding to a transaction entered at market conditions. The city has ordered other valuations from other institutions. One study (from a top-four consulting firm) has shown the market value at the trade date to be even CZK -274 million, another (from a private economic university) just said that it was really difficult to determine any market value, and another unofficial indicative valuation provided by a bank came up with the market value of -194 million. The first two valuations (CZK -262 million and CZK -274 million) were based on the principle where the future unknown swap rates are replaced by the forward swap rates implied by the term structure of interest rates valid at the valuation date. The same technique with a similar result (CZK -280 million) is used for example in the textbook on derivatives by Jílek (2006) where the swap is valued in detail. We will denote this approach (*i.e.* straightforward replacement of future unknown rates with the forward implied rates) as the Valuation Method No. 1. The method of the third valuation (CZK -194 million) has not been publicly disclosed in detail.

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We will use this specific transaction as a case study to illustrate that the straightforward rate replacement method is in fact incorrect, though not too far from a precise analytic valuation that we shall obtain and that will lie somewhere between the valuations mentioned above.

3. Derivatives Market Value

It is generally assumed that every derivative has a uniquely determined market value at any time from its inception to the final settlement date. International Accounting Principles (IAS 39) require that the fair (market) value of derivatives is regularly accounted for in the balance sheet and/or profit loss statement. The principles, however, do not say how the fair value should be exactly calculated in specific cases.

The market value of a derivative can be observed if there is a liquid market where the contractual rights and obligations are transferred from one counterparty to another for a price that is publicly quoted. This is essentially only the case of exchange-traded futures and options. Exchange traded futures (including their prices) are reset daily together with daily profit loss settlement on a margin account. The cumulative profit loss can be considered as the market value of the original futures position. On the other hand options are traded for their market premium representing the actual observable market value.

The market value cannot be directly observed for Over-the-Counter (OTC) derivatives that are generally not transferable and in many cases are entered with specific parameters that make comparison to other transactions difficult. Some OTC derivatives can, however, be reduced using a few elementary operations to a fixed cash flow and its present value then can be taken as the correct market value (disregarding counterparty credit risk). The traders sometimes call these types of derivatives "*plain vanilla*". More complex OTC derivatives with a liquid market can be also compared during their life to other quoted instruments that usually allow reducing the outstanding transaction to a fixed cash flow. As any new transaction conducted at market conditions has its market value close to zero, the present value of the difference cash flow is then a good estimation of the market value. Hence, the biggest problem is posed by derivatives that are not plain vanilla and lack a liquid standardized market like our case study exotic swap. There is a philosophical question what the right method for valuation of such exotic transactions is.

To show that derivatives involving Libor or swap rates in arrears are not in fact plain vanilla we firstly need to introduce the notion more formally. As we said in the introduction, we will restrict ourselves to derivatives that can be defined as finite sequences of payments at specified times where each payment is determined as a function of market variables observed at or before the time of each of the payments. Formally, each single cash flow can be expressed as $\mathbf{C} = \langle C, Curr, T \rangle$ where T is the time of the payment $C = f(V_1(t_1),...,V_n(t_n))$ in the currency $Curr$, the values $V_1(t_1),...,V_n(t_n)$ are observed market variables (asset prices, interest rates, foreign exchange rates, credit spreads, equity indexes, *etc.*) or other indices (weather, insurance, *etc.*) at times $t_i \leq T$, and f is a function. Forward Rate Agreements or European options can be defined in this way by a single payment. Financial derivatives with more payments like swaps can be formally defined as $D = \{\mathbf{C_1},...,\mathbf{C_m}\}$. Given two derivatives $D_1 = \{\mathbf{C_1},...,\mathbf{C_m}\}$

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and $D_2 = \{\mathbf{Q_1},...,\mathbf{Q_m}\}$ it is useful to define the sum $D_1 + D_2$ in a natural way as the sequence of the cash flows C_i and Q_j, or $C_i + Q_j$ in the case when the payment times coincide.

When valuing the derivatives we take the usual assumption of being in an idealized financial world where all financial assets can be traded, borrowed, and lent with perfect liquidity, without any spreads, taxes, or transaction costs, and where arbitrage opportunities do not exist. We will use risk-free interest rates $R(Curr,t)$ in continuous compounding for maturity t in the currency $Curr$. Normally we drop the parameter $Curr$ as we will focus mostly on single (domestic) currency derivatives.

A number of derivatives can be valued using the following three elementary principles:

(3.1) If $D = \{\mathbf{C_1},...,\mathbf{C_m}\}$ is a derivative consisting of fixed payments at $T_1 < \cdots < T_m$ then the market value

$$MV(D) = \sum_{i=1}^{m} e^{-R(T_i)\cdot T_i} \cdot C_i .$$

(3.2) If $D = \{\mathbf{C_1},\mathbf{C_2}\}$ is a pair of cash flows representing a money market deposit transaction, *i.e.* the principal C_1 is determined at T_1 and C_2 equals to $-C_1$ and the accrued market interest set at T_1 (in practice usually two business days before) for the period lasting from T_1 to T_2, then

MV(D) = 0.

(3.3) If $D_1,...,D_n$ are derivatives and $D = D_1 + \cdots + D_n$ then

$$MV(D) = \sum_{i=1}^{n} MV(D_i).$$

The three principles are already sufficient to value a number of "*plain vanilla*" interest rate derivatives. It is straightforward to generalize the principles in order to value simple derivatives with other underlying assets.

For example a $T_1 \times T_2$ forward rate agreement contract paying a fixed rate R_{FRA} on a nominal N is represented by the single cash flow

$$C = N \cdot \frac{(R_{FRA} - R_M)\cdot \tau}{1 + R_M \cdot \tau} \quad ,$$

where R_M is the reference market rate (usually Libor) observed at T_1 for the period from T_1 to T_2 and τ is the time factor calculated in an appropriate day-count convention. The derivative $D_0 = \{\mathbf{C}\}$ with single-variable cash flow settled at T_1 can be transformed using the principles (3.2) and (3.3) to a fixed cash flow. While we let the upper case R denote in general an interest rate *p.a.* to simplify our formulas we will sometimes use the lower case $r = R\cdot\tau$ for the time adjusted interest rates. Set

$D_1 = \{-C, C\cdot(1 + r_M)\}$ and
$D_2 = \{-N, N\cdot(1 + r_M)\}$ in both cases the cash flows paid at T_1 and T_2.

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Then $D_0 + D_1 = \{N \cdot (r_{FRA} - r_M)\}$ is paid at T_2 and

$D = D_0 + D_1 + D_2 = \{-N, N \cdot (1 + r_{FRA})\}$ is a fixed cash flow at T_1 and T_2.

The transactions D_1 and D_2 are both of the type (3.2), *i.e.* money market deposits, hence

$$MV(D_1) = MV(D_2) = 0 \text{ and so}$$

$$MV(D_0) = MV(D) = N \cdot (1 + R_{FRA} \cdot \tau) \cdot e^{-R(T_2) \cdot T_2} - N \cdot e^{-R(T_1) \cdot T_1}.$$

Thus the market value of an FRA equals to zero if the interest rate R_{FRA} equals to the forward rate implied by the current yield curve:

$$R_{FRA} = \frac{1}{\tau}(e^{R(T_2) \cdot T_2 - R(T_1) \cdot T_1} - 1). \tag{3.4}$$

A plain vanilla interest rate swap contract I_0 receiving the fix and paying the float interests paid in the same periods and the same day-count convention can be defined as a series of cash flows $C_i = (R_{fix} - R_M) \cdot N \cdot \tau(T_{i-1}, T_i)$ paid at T_i with R_M being the reference market rate observed at T_{i-1} for the period $[T_{i-1}, T_i]$. It turns out that using the FRA contracts the IRS cash flow can be transformed to a fixed cash flow. For each C_i it is sufficient to use the $T_{i-1} \times T_i$ FRA with the same nominal in the form $F_i = D_0 + D_1 = \{N \cdot (R_M - R_{FRA}) \cdot \tau\}$ paid at T_i as above. Then the modified cash flow of $I_0 + F_i$ paid at T_i is fixed as

$$(R_{fix} - R_M) \cdot N \cdot \tau + (R_M - R_{FRA}) \cdot N \cdot \tau = (R_{fix} - R_{FRA}) \cdot N \cdot \tau \ .$$

Hence if I_0 is a plain vanilla IRS then $MV(I_0) = MV(I)$ where $I = I_0 + F_1 + \cdots + F_n$ is a combination of the original swap and a series of FRAs for each float interest payment. As the FRA interest rates are applied at market conditions we have $MV(F_1) = \cdots = MV(F_n) = 0$. The cash flow I results from I_0 replacing the unknown float payments by forward interest rates implied by the current term structure.

Similarly it can be shown that the original IRS cash flow can be transformed using the principles (3.1)-(3.3) to the fixed cash flow paying the first fixed float interest plus the nominal N at T_1 and on the other hand receiving the fix interest payments plus the nominal N at maturity T_n. Thus at the start date of any market IRS transaction the following must hold:

$$1 = \sum_{i=1}^{n} R_{fix} \cdot \tau(T_{i-1}, T_i) \cdot e^{-R(T_i) \cdot T_i} + e^{-R(T_n) \cdot T_n}, \text{ i.e.}$$

$$R_{fix} = \frac{1 - e^{-R(T_n) \cdot T_n}}{\sum_{i=1}^{n} \tau(T_{i-1}, T_i) \cdot e^{-R(T_i) \cdot T_i}}. \tag{3.5}$$

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4. Plain Vanilla Derivatives

In both cases given an FRA or IRS derivative transaction D in the previous section we have in fact found a finite number of elementary derivatives $D_1,...,D_m$ of the type (3.1) or (3.2) so that $D_1+\cdots+D_m$ is a replication of D, *i.e.* $D=D_1+\cdots+D_m$.

Definition: A derivative D will be called *linear plain vanilla* if it can be replicated at its start date as $D_1+\cdots+D_m$ where $D_1,...,D_m$ are of the type (3.1) or (3.2).

We are using the adjective "*linear*" since ordinary European options (with a nonlinear payoff) cannot be certainly statically replicated using transactions of the type (3.1) or (3.2) but are still often called "*plain vanilla*".

We ask the question how broad is the class of linear plain vanilla derivatives. Besides the FRA and IRS does it also contain other swaps like swaps with Libor or swap rates in arrears? Note that the operations of type (3.2) allow moving even a future interest payment forward and backward so the positive answer cannot be simply ruled out. To find market values of *swaps with Libor in arrears* (see *e.g.* Li, Raghavan, 1996) it is sufficient and necessary to value in general the cash flow $C=r_M(T_1,T_2)=R_M(T_1,T_2)\cdot\tau(T_1,T_2)$ payable at T_1 (instead of T_2) where $R_M(T_1,T_2)$ is the market rate (Libor) observed at T_1 for the period lasting from T_1 to T_2. Notice that if r_M was discounted to $r_M/(1+r_M)$ then the cash flow could be moved using an operation of type (3.2) to the ordinary time T_2 and valued in the same fashion as in the case of FRA, *i.e.* replaced with the forward rate and discounted to time zero. We will show elementarily that the missing discount factor $1/(1+r_M)$ in the cash flow C is in fact essential.

Proposition 1: The Libor in arrears $L=\{\langle r_M(T_1,T_2),T_1\rangle\}$ is not a linear plain vanilla derivative.

Proof: Assume by contradiction that L can be expressed as a sum of derivatives of the type (3.1) or (3.2). Since any sum of fixed cash flows of type (3.1) is again a fixed cash flow we can assume that $L=F+P_1+\cdots+P_n$ is a sum of one fixed cash flow F and finitely many cash flow pairs $\{\mathbf{C_1},\mathbf{C_2}\}$ of the type (3.2). Recall that by definition $\mathbf{C_1}=\langle C_1,t_1\rangle$ can be any cash flow determined by a function of information available at time t_1 and $C_2=C_1(1+r_M(t_1,t_2))$ payable at t_2 is equal to C_1 plus the accrued market interest observed at t_1. Consequently $r_M(T_1,T_2)$ must be in the form

$$(4.1)\qquad r_M(T_1,T_2)=a+\sum_{i=1}^{m}b_i\cdot(1+r_M(t_i,T_1))+\sum_{j=1}^{k}c_j\,,$$

where a is a constant amount known at time 0, $t_1<\cdots<t_m<T_1$, nominal amounts b_i are determined at t_i, and c_j are nominal amounts of pairs (deposits) starting at T_1. Note that the first two parts on the right hand side of are determined at or before t_m, hence the sum of c_j equals to $r_M(T_1,T_2)-A$ where A is a value already determined at the time $t_m<T_1$. The equation $L=F+P_1+\cdots+P_n$ must hold for all interest rate scenarios

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so we may restrict ourselves to the scenarios where all rates $r_M(t_1,t_2)$ for $T_1 < t_1 < t_2$ are forward implied by the rates $r_M(T_1,t)$ for $t > T_1$. Under this assumption when all the rates from T_1 on can be compounded it is easy to show that for any pair P of type (3.2) with payment times $T_1 \le t_1 < t_2 < T_M$ we can find two pairs Q_1 and Q_2 of the type (3.2) with payment times at t_1, T_M and t_2, T_M respectively so that $P = Q_1 + Q_2$. Consequently with respect to this set of interest rate scenarios we may decompose all P_i with the first payment time $t_i > T_1$ and the second being some large T_M. Hence we may assume without loss of generality that for all such P_i the second payment time is some fixed $T_M > T_2$. Consider such a pair with the first payment at $t_i > T_1$. We may certainly assume that there is only one pair $P_i = \{-C_i, C_i(1 + r_M(t_i, T_M))\}$ with payment times t_i and T_M but in addition there could be other pairs P_k with the first payment time $t_k < T_1$ and the second at the t_i. The sum of all the cash flows at t_i in the decomposition $L = F + P_1 + \cdots + P_n$ must be identically zero hence it follows that the C_i is a value determined already before the time T_1. Moreover the cash flow $\sum c_j = r_M(T_1,T_2) - A$ is now also discounted to T_M. Finally the cash flow at T_M must be also identically equal to zero:

$$0 = \sum C_i \cdot (1 + r_M(t_i, T_M)) + (r_M(T_1, T_2) - A) \cdot (1 + r_m(T_1, T_M)) + D \,. \tag{4.2}$$

The first sum in (4.2) is taken over all P_i with the first payment at $t_i > T_1$ and the second at T_M. The second expression corresponds to the pair with payment times at T_1, T_M, and D is the sum of a constant payment and of all the final payments from pairs starting at some $t < T_1$ and ending at T_M. This equation cannot clearly hold as the values $C_i, A,$ and D have been determined before T_1 and after that time we admit *e.g.* all the interest rate scenarios with $r_M(t_1,t_2) = e^{R(t_2 - t_1)} - 1$ for arbitrary $R > 0$ and $t_2 > t_1 \ge T_1$.

Note that the equation (4.2) could hold if the cash flow $r_M(T_1,T_2)$ at T_1 was replaced with the discounted interest rate $r_{M,disc} = r_M(T_1,T_2) / (1 + r_M(T_1,T_2))$, for example

$$1 + (r_{M,disc} - 1)(1 + r_M) = 0 \,.$$

A *constant maturity swap* is a swap where one counterparty pays to the other fixed interest rate and the other pays the swap rate with a constant maturity M observed always at the time (or right before) of any payment. Again to value constant maturity swaps it is necessary and sufficient to value a single swap rate in arrears payment $\langle s_T, T \rangle$ where the market rate is determined at T for interest rate swaps with maturity M. Here we assume a liquid IRS market so that the reference rate s_T follows the equation(3.5). Similarly to swaps with Libor in arrears we hypothesize that constant maturity swaps are not (linear) plain vanilla.

One may want to extend the type (3.2) operations with cash flows corresponding to swaps starting at T, ending at $T + M$, and with the market swap rate s_M observed at T for that maturity. However s_M is by the equation (3.5) a function of the interest rates known at the time T and so the swap cash flow can be replicated as combination of the elementary operations of the type (3.1) and (3.2).

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Proposition 2: The swap rate in arrears $S = \{\langle s_T, T\rangle\}$ is not a linear plain vanilla derivative.

Proof: If by contradiction $S = F + P_1 + \cdots + P_n$ then we may use the same argumentation as in the proof above ending up with the equation

$$0 = \sum_{t_i > T} C_i \cdot (1 + r_M(t_i, T_M)) + (s_T - A) \cdot (1 + r_M(T, T_M)) + D \,. \tag{4.3}$$

This equation cannot hold in all scenarios when the instantaneous interest rate is set to an arbitrary $R > 0$ from the time T on, so that $s_T \approx R$, $r_M(t, T_M) = e^{R(T_M - t)} - 1$, and the values C_i, A, and D have been determined before T and so are independent on R.

5. Expected Value Principle

Even though we have proved that the swaps with float rates in arrears cannot be replicated in a straightforward way we might still try to use the Expected Value Principle to show that the future unknown interest rates may be replaced with the forward rates and discounted to time 0 with the risk-free interest rates.

The Expected Value Principle or rather the Risk Neutral Valuation Principle says that if V_t is the value of a derivative at time t with payoff V_T paid at time T and determined as a function of prices some underlying assets then

$$V_0 = P(0,T) E_T[V_T], \tag{5.1}$$

where the expectation is taken in the world that is forward risk neutral with respect to the $P(t,T)$, *i.e.* time t value of a unit zero coupon bond maturing at T (see for example Hull, 2006, or Hunt, Kennedy, 2000). An ingenious argument proving the principle is also based on the replication principle, however, in infinitesimally small time intervals and dynamically readjusted. It has been used first by Black and Scholes (1973) to value stock options under the assumptions of constant or at least deterministic interest rates. This assumption must be relaxed in order to value interest rate derivatives. This can be achieved using the value of the money market account or $P(t,T)$ as the numeraire. For any numeraire g there is a measure so that for any derivative f with the same source of uncertainty the process $\frac{f}{g}$ is a martingale, in particular $\frac{f(0)}{g(0)} = E_T\left[\frac{f(T)}{g(T)}\right]$ (see Málek, 2005 or Harrison, Pliska, 1981). The measure (or the world) is called forward risk-neutral with respect to the numeraire g. Specifically if $g = P(t,T)$ then

$$\frac{V_0}{P(0,T)} = E_T\left[\frac{V_T}{P(T,T)}\right],$$

which implies (5.1) as $P(T,T) = 1$. The equation holds for all derivatives, including those that depend on interest rates. The world is risk neutral with respect to $P(t,T)$ if the return of any asset from t to T equals to the return of risk free zero coupon bonds maturing at T. On the other hand, if we set g equal to the value of the money market

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account then $g(t) = \exp(\int_0^t R(s)ds)$ and it follows

$$V_0 = \hat{E}_T\left[-\exp(\int_0^T R(s)ds)\cdot V_T\right], \tag{5.2}$$

where the expectation is taken in the world that is forward risk neutral with respect to the money market account. In this world the return of any asset in a time period equals to the return of the money market account.

Going back to the issue of valuation of swaps with rates in arrears we prefer the equation (5.1) where the discounting is taken out of the expectation operator (see also Pelsser, 2003, Musiela, Rutkovski, 1997, or Gatarek, 2003). The idea to replace the future unknown rates with the forward ones would still be applicable if we were able to show that the expected value $E_T[r_M(T,T')]$ equals to the forward rate. However, it follows that there is a difference between the two values, the former being greater than the latter, and so an adjustment is needed if the forward rates are to be used as a proxy of the expected values.

6. Convexity Adjustments

Estimating the expected value of a Libor in arrears $E_T[r_M(T,T')]$ one has to realize that an interest rate itself is not a tradable asset. If A_t denotes the time t price of a tradable asset (paying no income and with zero storage cost) then its time 0 non-arbitrage forward price $A_T^F(0)$ must solve the equation $P(0,T)E_T[A_T^F(0) - A_T] = 0$, hence

$$A_T^F(0) = E_T[A_T] \tag{6.1}$$

If we set $A_T = P(T,T')$ then $r_M(T,T') = \frac{1-A_T}{A_T} = \frac{1}{A_T} - 1$ is a nonlinear function of A_T. Recall that in general if g is a strictly convex function and X a non-trivial random variable (*i.e.* not attaining only one value with probability 1) on a probability space then by Jensen's inequality $g(E[X]) < E[g(X)]$. Since $g(X) = \frac{1}{X} - 1$ is strictly convex for $X > 0$ and the random variable $A_T > 0$ is nontrivial we get

$$r_F(0,T,T') = \frac{1-A_T^F(0)}{A_T^F(0)} = g\left(E_T[A_T]\right) < E_T[g(A_T)] = E_T[r_M(T,T')]. \tag{6.2}$$

The difference between the right hand side and left hand side of the strict inequality is the convexity adjustment that we need to calculate or at least estimate if the forward rate $r_F(0,T,T')$ is to be used as a proxy for $E_T[r_M(T,T')]$.

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On the other hand note that if the interest rate $r_M(T,T')$ is payable at T' and if we use $P(t,T')$ as the numeraire, then $r_F(t,T,T') = \dfrac{P(t,T)-P(t,T')}{P(t,T')}$ is a martingale and so

$$r_F(0,T,T') = E_{T'}\left[r_F(T,T,T')\right] = E_{T'}\left[r_M(T,T')\right].$$

One popular way to estimate the convexity adjustment discovered by Brotherton-Ratcliffe, Iben (1993) is to use the Taylor expansion of the inverse function $f = g^{-1}$. If $A = f(r)$ with $r = r_M$ and dropping the parameters T and T' we have

$$A - A^F = f(r) - f(r^F) = f'(r_F)(r - r_F) + \frac{1}{2} f''(r_F)(r - r_F)^2 + \cdots$$

Neglecting the terms of the third and higher order and applying the expectation operator we get

$$0 = E\left[A\right] - A^F \cong f'(r_F)(E\left[r\right] - r_F) + \frac{1}{2} f''(r_F) \cdot E\left[(r - r_F)^2\right] \text{ and so}$$

$$E\left[r\right] - r_F \cong -\frac{1}{2} \cdot w_r^2 \frac{f''(r_F)}{f'(r_F)}, \tag{6.3}$$

where we have used one more approximation $w_r^2 = Var(r) \cong E\left[(r - r_F)^2\right]$. In the case of Libor in arrears $f(r) = \dfrac{1}{1+r}$ the convexity adjustment estimation takes the simple form $\dfrac{w_r^2}{1+r_F}$. Normally w_r^2 is expressed as $\sigma_r^2 \cdot T \cdot r_F^2$ where σ_r is an estimation of the volatility of the Libor rate from historical data. The formula has been also extended by Benhamou (2000a, 2000b) in the framework of time dependent deterministic volatility. It seems that performance of the convexity adjustment estimation might be simply improved if the Taylor expansion was applied directly to the function $g(A)$ (see Henrard, 2007). However we will get a closed formula under the assumption of log-normality of A at the end of this section.

Popular Convexity Adjustment Formula for Swap Rates in Arrears

Regarding swap rate in arrears we need to find $E_T\left[s\right]$ where $s = s_M(T, T+M)$ is the market swap rate observed at T for swaps of length M and the expectation is taken in the world that is again forward risk neutral with respect to $P(t,T)$.

If $P = P(T, T+M)$ and $A = \sum_{i=1}^{m} P(T,T_i) \cdot \tau_i$, where $T_1, ..., T_m = T + M$ are the fixed interest rate payment times and $\tau_i = \tau_i(T_i, T_{i+1})$ the time adjustment factors, then according to (3.5) the swap rate $s = g(P,A) = \dfrac{1-P}{A}$. P and A are prices of tradable assets at time T.

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(A corresponding to a portfolio of zero coupon bonds) and so according to (6.1) the forward prices of P and A at the time T calculated at $t = 0$ equal to their expected value in the world that is $P(t,T)$ forward risk neutral: $P^F = E_T[P]$, $A^F = E_T[A]$. Since the function $g(P,A)$ is strictly convex in A analogously to we get the inequality $s_F < E_T[s]$. To get a simple convexity adjustment formula in the style of (6.3) we need to condense the two variables into one. According to Hull (2006) let B be the market price quoted at time T of the bond with maturity at $T + M$, unit nominal value, and fixed coupon rate s_F paid at $T_1,...,T_m$. If y is the market yield of the bond then $B = f(y)$ and as s is a proxy of y we can use the approximation $B \cong f(s)$. Applying (6.3) we obtain

$$E_T[s] - s_F \cong -\frac{1}{2} \cdot w_s^2 \frac{f''(s_F)}{f'(s_F)} = \frac{1}{2}\sigma_s^2 s_0^2 T \frac{C}{D}, \tag{6.4}$$

where w_s is the standard deviation of s_T, σ_s the volatility of s, C convexity, and D the duration of the bond at $y = s_F$. When this formula is used for valuation of a constant maturity swap we will call the approach Valuation Method No. 2.

Modified Single-variable Taylor Expansion Based Convexity Adjustment

An alternative approach is to consider directly the swap rate to be a function of the bond price, $s = g(B)$. Taking the Taylor expansion of the function at B^F we get

$$s - s_F = g(B) - g(B^F) = g'(B^F)(B - B_F) + \frac{1}{2}g''(B^F)(B - B^F)^2 + \cdots$$

Now let us apply the expectation operator and the fact that $E_T[B] = B^F$ to derive hopefully a little bit more precise convexity adjustment formula

$$E_T[s] - s_F \cong \frac{1}{2}g''(B^F)E_T\left[(B - B^F)^2\right] = -\frac{1}{2}w_B^2 \frac{f''(s_F)}{\left(f'(s_F)\right)^3}. \tag{6.5}$$

The formula is consistent with (6.4) as $w_B \cong w_s \cdot f'(s_F)$ however in derivation of (6.4) we have taken one more approximation step compared to (6.5). Consequently we expect this formula to lead to a better valuation of a given constant maturity swap that we will call Valuation Method No. 3.

Two-Variable Taylor Expansion Based Convexity Adjustment

The estimation can be further improved if we return to the two-variable function expressing the swap rate, $s = g(P,A) = \frac{1-P}{A}$. Let us expand again the difference $s - s_F$ using the Taylor formula

$$s - s_F = g_P(P^F, A^F)(P - P^F) + g_A(P^F, A^F)(A - A^F) + \frac{1}{2}g_{PP}(P^F, A^F)(P - P^F)^2 +$$
$$+\frac{1}{2}g_{AA}(P^F, A^F)(A - A^F)^2 + g_{PA}(P^F, A^F)(P - P^F)(A - A^F) + \cdots$$

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Neglecting the third and higher order terms and taking the expectation we get

$$(6.6) \qquad E_T[s] - s_F \cong \frac{1}{2} g_{P} \cdot w_P^2 + \frac{1}{2} g_{A} \, w_A^2 + g_{PA} \cdot Cov_{P,A},$$

where the partial derivatives are taken at the forward values P^F and A^F. Applying the formula on $s = g(P,A) = \frac{1-P}{A}$ we finally get

$$(6.7) \qquad E_T[s] - s_F \cong \frac{1-P^F}{\left(A^F\right)^3} w_A^2 + \frac{\rho \cdot w_P \cdot w_A}{\left(A^F\right)^2}.$$

Compared to (6.5) we have eliminated one more approximation step and derived a presumably better convexity adjustment formula that need to estimate not only volatilities of the prices P and A but also their correlation ρ. Pricing of constant maturity swaps obtained using the formula (6.7) will be the called Valuation Method No. 4.

Multi-Lognormal-Variable Based Valuation Formula

Finally we will use the particular form of the function $g(P,A)$ to derive a closed formula for $E_T[g(P,A)]$ under the assumption of lognormality of the variables P and A. Recall (see *e.g.* Aitchinson and Brown, 1996) that if X is a lognormally distributed random variable and $\ln\left(\frac{X}{X_0}\right) \approx N(m,s)$ is normal with mean m and standard deviation s then the expected value $E[X] = X_0 e^{m+s^2/2}$ and the variance $\sigma^2(X) = X_0^2 e^{2m}\left(e^{s^2} - 1\right)$.

Let us assume that $P = P(0,T,T+M)$ and $A = A(0,T,T+M)$ are jointly lognormally distributed in the measure that is forward risk neutral with respect to $P(t,T)$:

$$\ln\left(\frac{P}{P^F}\right) \approx N\left(-\frac{\sigma_P^2}{2}T, \sigma_P\sqrt{T}\right),$$

$$\ln\left(\frac{A}{A^F}\right) \approx N\left(-\frac{\sigma_A^2}{2}T, \sigma_A\sqrt{T}\right),$$

so that $E[P] = P^F$ and $E[A] = A^F$. If P and A are lognormal then clearly $\frac{1}{A}$ and $\frac{P}{A}$ are lognormal as well since

$$\ln\left(\frac{1/A}{1/A^F}\right) = -\ln\left(\frac{A}{A^F}\right) \approx N\left(\frac{\sigma_A^2}{2}T, \sigma_A\sqrt{T}\right) \text{ and}$$

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$$\ln\left(\frac{P/A}{P^F/A^F}\right)=\ln\left(\frac{P}{P^F}\right)-\ln\left(\frac{A}{A^F}\right)\approx N\left(\left(\frac{\sigma_A^2}{2}-\frac{\sigma_P^2}{2}\right)T,\sqrt{\sigma_A^2-2\rho_{A\sigma_P}+\sigma_P^2}\sqrt{T}\right).$$

Using the relationship between the expected value and volatility of a lognormal variable and its exponential power we get

$$E_T\left[\frac{1-A}{P}\right]=E_T\left[\frac{1}{A}\right]-E_T\left[\frac{P}{A}\right]=\frac{e^{\sigma_A^2T/2+\sigma_A^2T/2}}{A^F}-\frac{P^F}{A^F}e^{(\sigma_A^2-\sigma_P^2+\sigma_A^2-2\rho_{A\sigma_P}+\sigma_P^2)\cdot T/2}=$$
$$=\frac{e^{\sigma_A^2T}-P^F\cdot e^{(\sigma_A^2-2\rho_{A\sigma_P})T}}{A^F}$$

Consequently under the lognormality assumption the precise formula for the convexity adjustment is

$$E_T[s]-s^F=\frac{\left(e^{\sigma_A^2T}-1\right)-P^F\cdot\left(e^{(\sigma_A^2-2\rho_{A\sigma_P})T}-1\right)}{A^F}. \tag{5.8}$$

To apply the formula we need to estimate the stochastic volatilities σ_P,σ_A, and the correlation ρ. The valuation of a constant maturity swap using the formula will be called Valuation Method No. 5.

The approach could be generalized to derivatives with payoff of the form

$$g(A_1,...,A_k)=\frac{p(A_1,...,A_k)}{A_1\cdots A_k},$$

where $A_1,...,A_k$ are prices of some underlying assets observed at or before the payoff time T and p is a polynomial. If we assume that $A_1,...,A_k$ are jointly lognormally distributed then g can be similarly decomposed into a sum of lognormal variables. The expected value of each part and the sum can be then expressed in an analogous way as above.

Example: Consider for example a Quanto Pribor in Arrears derivative denominated in CZK and paying in one year the actual 1Y Pribor in arrears multiplied by the annual appreciation of EUR with respect to USD. Similar products do appear in the market. The payoff can be expressed as

$$g(P,E,U)=\frac{1-P}{P}\cdot\frac{E}{U}\cdot\frac{U_0}{E_0}=\frac{(1-P)E}{PU}\cdot\frac{U_0}{E_0}=\left(\frac{E}{PU}-\frac{E}{U}\right)\cdot\frac{U_0}{E_0}$$

where $P(1,2)$ is the value of a one-year-to-maturity zero coupon bond at year 1, E the exchange rate of EUR in CZK, U the exchange rate of USD in CZK at year 1, and E_0, U_0 the initial exchange rates. If P,E and U are jointly lognormally distributed with respect to the measure that is forward risk neutral to $P(t,1)$ then as above $\frac{E}{PU}$

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and $\frac{E}{U}$ are lognormal and we can express $E\left[\frac{E}{PU}-\frac{E}{U}\right]$ in terms of the volatilities and correlations of *P, E* and *U*. To simplify the calculation we may set $S=\frac{E}{U}$ that is also lognormal in the chosen measure. If

$$\ln\left(\frac{P}{P^F}\right)\approx N(-\frac{\sigma_P^2}{2},\sigma_P)\ ,$$

$$\ln\left(\frac{S}{S^F}\right)\approx N(-\frac{\sigma_S^2}{2},\sigma_S)\ ,$$

so that $E[P]=P^F$ and $E[S]=S^F$. Then assuming joint lognormality of P and S we get:

$$\ln\left(\frac{S/P}{S^F/P^F}\right)=\ln\left(\frac{S}{S^F}\right)-\ln\left(\frac{P}{P^F}\right)\approx N\left(\left(\frac{\sigma_S^2}{2}-\frac{\sigma_P^2}{2}\right),\sqrt{\sigma_S^2-2\rho_{S\sigma_P}+\sigma_P^2}\right), \text{ and}$$

$$E_T\left[\left(\frac{S}{P}-S\right)\frac{1}{S_0}\right]=\left(E_T\left[\frac{S}{P}\right]-E_T[S]\right)\frac{1}{S_0}=\left(\frac{S^F}{P^F}e^{(\sigma_S^2-\sigma_P^2+\sigma_S^2-2\rho_{S\sigma_P}+\sigma_P^2)/2}-S^F\right)\frac{1}{S_0}=$$

$$=\frac{S^F}{S_0}\left(\frac{e^{\sigma_S^2-2\rho_{S\sigma_P}}}{P^F}-1\right)=e^{R_{USD}-R_{EUR}}\left(\frac{e^{\sigma_S^2-2\rho_{S\sigma_P}}}{P^F}-1\right)$$

where R_{USD}, R_{EUR} are one year interest rates in the two currencies and ρ is the correlation between $S=S_{EUR/USD}$ and $P=P_{CZK}(1,2)$.

7. Valuation of the Case Study Exotic Swap

We have identified five possible methods for valuation of swaps involving swap rates in arrears like the one described in Section 2. The methods may be summarized as follows:

1. Replace the future unknown rates with the forward rates implied by the current yield curve without any adjustment and discount the resulting cash flow forecast.
2. Add an adjustment based on volatilities of the swap rates using the formula (6.2).
3. Add an adjustment based on volatilities of bonds with coupons set at the level of the forward swap rates using the formula (6.3).
4. Add an adjustment based on a more precise formula (6.7) involving volatilities and correlations of zero coupon bonds and annuities.
5. Calculate the expected swap rates using the closed formula (6.8) based on volatilities and correlations of zero coupon bonds and annuities.

We have performed the valuation with market data as of March 12, 2003. To apply the Valuation Method No. 1 we have used the same swap rates as some of the consulting firms mentioned in Section 2 (see Table 2). The used EUR/CZK exchange rate is 31,665.

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Table 2
The Swap Rates Used in the Valuation Method No. 1

Maturity	1	2	3	4	5	6	7	8	9	10	12	15	20
EUR	2,35%	2,42%	2,67%	2,93%	3,18%	3,39%	3,57%	3,74%	3,88%	4,00%	4,00%	4,00%	4,00%
CZK	2,18%	2,32%	2,55%	2,79%	3,04%	3,26%	3,46%	3,64%	3,79%	3,91%	4,10%	4,29%	4,44%

The swap rates are available up to 20 years maturity and so the mid rates can be used for a relatively precise construction of the discount rates, and forward rates up to the maturity date of the swap. But to calculate convexity adjustments using the Methods No. 2–5 we need to plug in certain volatilities, or even correlations of the underlying assets. It would be optimal if we could use market quoted forward-looking volatilities on bond options, swaptions, caps, or caplets. However the market with interest rate derivatives has not been sufficiently developed so far (see *e.g.* Vojtek, 2004 or Mičulka, 2007) and all we can do is to use historical data to make certain estimations. Again we could use a number of different methods leading to a multitude of slightly different results in each of the approaches 2-5. The estimations may be based on different lengths of the historical data, may use different weights, different assumptions on the stochastic processes *etc*.

We have used historical swap rates provided by Reuters that start in the case of CZK in 1998. The quality of data is not very good (missing time periods) until 2000 due to low liquidity and the financial crisis in late nineties. This is a reason to take only a shorter history of equally weighted data.

Another key issue is lack of historical swap rate quotes with maturities beyond 10 years before 2004. For example to estimate the standard deviation of the market value of the 10 year annuity $A = A(5,15)$ starting in 5 years and maturing in 15 years (March 12, 2003 corresponds to $t = 0$) observed in 5 years we could use essentially two basic approaches. One would be just to calculate the historical volatility of $A(0,10)$. However, this approach clearly underestimates the standard deviation of *A(5,15)* since we are modelling volatility of the price of a fixed cash flow maturing 15 years from now hence its volatility will be definitely higher at the beginning than at the end of the modelled 5 years period. Another possibility is to model the process for the present value of the annuity calculated with the interest rates known at time t, $A(t) = A(t,5,15)$, as $dA = \mu A dt + \sigma A dz$ with a positive drift μ and a volatility σ that is not constant. To eliminate the drift we will rather replace $A(t)$ with the forward value of the annuity calculated at time t, *i.e.* set $A(t) = A^F(t,5,15)$. The volatility then still depends on the time t (empirically it is decreasing with t as there is less uncertainty with a shorter time $15 - t$ to maturity of the observed instrument) and must be estimated at least for the years 1 to 5 taking the quadratic average volatility as the input into the convexity adjustment formula. So we may use the historical data to estimate the volatilities $\sigma_1,...,\sigma_5$ of prices of $A^F(5,15)$, $A^F(4,14)$,…,and $A^F(1,11)$. The estimation of σ_A then will be $\sqrt{\left(\sigma_1^2 + \cdots + \sigma_5^2\right)/5}$. To calculate historical prices of $A^F(10,20)$ we need to extent the yield curve up to 20 years maturity. The standard way to do this is to assume that the swap rates beyond 10 years are constant and equal to the 10 years swap rate. The extrapolation obviously significantly distorts the result but that is probably all we can say unless we apply a dynamic interest rate model as *e.g.*

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in Mercurio, Pallavicini (2006) or Hagan (2003), which could be subject of another study on the issue of interest rate model calibration in an emerging market with limited historical market data. Although there are many approaches we could use, we have decided to choose just one:

Use 300 business days historical mid swap rates quotations in CZK.

Extrapolate the rates beyond 10 years maturity with the 10 year swap rate.

Use just the historical 10 and 2 years maturity swap rates to estimate the volatilities of the future swap rates.

Use the data with equal weights to calculate historical volatilities of forward values of the cash flows (P, A and B) for individual years starting from the time zero to the float payment date. The final volatility estimation is then calculated as a quadratic average.

Correlations are calculated in the same way but taking a standard average instead of the quadratic one.

Table 3
The Market Valuations Using All Five Methods

Method	Adjustment (CZK million)	Market Value (CZK million)
1 - Forward Value Principle	0,000	-262,714
2 - Adjustment (6.2)	27,673	-235,041
3 - Adjustment (6.3)	22,663	-240,051
4 - Adjustment (6.7)	18,317	-244,397
5 - Adjustment (6.8)	18,356	-244,358

The market valuations applying the five methods shown in Table 3 indicate that the results differ but remain within the same order. The dispersion would be probably wider if we used also different volatility/correlation estimation methods. The popular convexity adjustment (2) seems, according to our analysis, to underestimate the most precise two-variable adjustments (4) and (5) while the improved single-variable adjustment (3) remains somewhere in between.

8. Conclusion

The paper has been motivated by a real life exotic swap transactions which was valued by financial practitioners in the range of CZK –194 to –280 million at the trade date of the transaction. Non-practitioners have assigned a positive value to the swap or even claimed that there is nothing like the trade date market value. International Accounting Standards require banking and non-banking subjects to account for the market value of derivatives on a regular basis and such dispersion of possible market values and opinions seems to be puzzling.

The first part of the paper rejected the hypothesis that swaps involving Libor or swap rates in arrears could be sort of "*plain vanilla*" derivatives, *i.e*. they cannot be replicated as a combination of elementary transactions like plain vanilla forward rate agreements or interest swaps. The proof is elementary and does not depend on any particular stochastic

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interest rate model. It follows that a convexity adjustment is needed, if the forward rates are to be used as a proxy for expected value of Libor or swap rates in arrears.

We have developed two improved convexity adjustment formulas, and a fully closed formula using a method applicable to a wide class of convexity related derivatives. Application of the formulas to the real life swap gave the results ranging from CZK –235 to –263 million with CZK –244 million identified as the most precise valuation. However, our analysis has shown that the result still remains in a mist with respect to the estimations of volatilities based on historical data from a not fully developed derivative market. The conclusion is that not only the case study swap was inappropriate for the City interest rate profile, but moreover it did present a significant risk in terms of the pricing uncertainty, that is due to existence of a number of complex and not always fully consistent models applied even by professionals, and due to lack of sufficient data on the underlying rates in the still developing market of CZK interest rate instruments.

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